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Gravitational Fields and Nonlinear σ-Models[1]

Jerónimo Cortez,[2] Dario Núñez,[2] and Hernando Quevedo[2]

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It is shown that different types of gravitational fields can be analyzed as nonlinear σ-models. We show that the Einstein–Hilbert action for stationary aximmetric fields, Einstein–Rosen gravitational wave, and Gowdy cosmological models can be expressed in terms of a Lagrangian density for the $SL(2, R)/SO(2)$ σ-model. We discuss the possibility of using these results to quantize gravitational fields.

1. INTRODUCTION

σ-Models were introduced in field theory as a way to construct Lagrangians in terms of scalar fields. The simplicity of such Lagrangians was especially useful to test different quantization procedures. Later, the geometric character of σ-models was noted (Macfarlane, 1979; Meetz, 1969; Salam *et al.*, 1969). First efforts were made to investigate nonlinear σ-models in flat two-dimensional space where they can be considered as a useful tool to perform theoretical experiments due to their affinities with four-dimensional gauge theories. In this context, it is possible to find classical solutions representing vacuum and nonvacuum configurations as well as to perform perturbative analysis for their quantization (D' Adda *et al.*, 1978, 1979; Duff *et al.*, 1977; Eichenherr, 1978; Golo *et al.*, 1978; Witten, 1979).

In higher dimensional flat spaces, nonlinear σ-models do not present such interesting features. For instance, in the four-dimensional case, there do not exist topologically nontrivial solutions. Nevertheless, different procedures can be invented to make such models interesting. For instance, one can couple them to other theories like gravity or gauge theories in Minkowski

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[2] Universidad Nacional Autónoma de México, Instituto de Ciencias Nucleares, Ciudad Universitaria, Circuito Exterior, Mexico City, Mexico; e-mail: cortez@nuclecu.unam.mx, nunes@-nuclecu.unam.mx, quevedo@nuclecu.unam.mx

space; in this case, the intrinsic topology allows one to find maps from the corresponging space-time to the space of fields of the nonlinear σ-model. Another possibility is to consider such models in two- or higher dimensional spaces with constant or nonconstant curvature where the intrinsic topology and the topology of the space of solutions is much richer than in flat space (de Alfaro *et al.*, 1979; Gava *et al.*, 1979; Quevedo, 1980).

Nonlinear σ-models consist of nonlinear scalar field theories with values in a coset space G/H which possesses the structure of a principal fiber bundle. Such a field theory is a nonlinear realization of the group G which becomes linear with respect to the subgroup H. In the usual formulation, the scalar fields of the action are redundant in the sense that they are subjected to nonlinear constraints and are characterized by a gauge freedom. However, it is possible to reexpress the action in terms of nonredundant fields, getting rid of the nonlinear constraints and the gauge freedom. In this case, the action becomes similar to the one used in the theory of harmonic maps (Misner, 1978). In several cases, it is even possible to express the Einstein–Hilbert action of general relativity as an action for a harmonic map and therefore the solutions of Einstein's equations can be interpreted as functional geodesics of the target space of the harmonic map (Núñez *et al.*, 1997).

The two-dimensional nonlinear σ-models are complete integrable; it is possible to find the infinite-dimensional symmetries of the underlying field equations wich allow one to integrate the partial differential equations and to find exact solutions by using recursive methods. This result has been used to extensively analyze the case of stationary axisymmetric gravitational fields (Hoenselaers and Dietz. 1984) and a class of exact solutions containing an infinite number of multipole moments (Quevedo, 1986). (Generalizations to the case of the Einstein–Maxwell theory coupled to a scalar field also have been investigated (Matos, 1994). These results are in the spirit of the well-known fact that upon dimensional reduction to two dimensions, which can be achieved, for instance, when the spacetime possesses two commuting Killing vector fields, Einstein–Maxwell theory is characterized by a group of non-Abelian symmetry transformations which act as Bäcklund transformations on the set of classical solutions. The integrability of the nonlinear σ-models has given a conceptually new insight into the problem of quantization of gravity. Recently, an exact quantization of axisymmetric stationary space-times has been proposed (Korotkin *et al.*, 1995), using the approach based on the complete separation of variables of the underlying σ-model. This new method allows one to treat exactly a quantized system with an infinite number of physical degrees of freedom.

The analogy between σ-models and gravitational fields has been treated so far at the level of the field equations, i.e., one shows that the equations derived from the Lagrangian of a σ-model are equivalent to the Einstein equations for a specific gravitational field. In this work, we show that this analogy holds also at the level of the Lagrangian density.

We derive the Lagrangian density for the $SL(2, R)/SO(2)$ nonlinear σ-model and show that it is equivalent to the reduced Einstein–Hilbert action for stationary axisymmetric fields, Einstein–Rosen gravitational waves, and Gowdy cosmological models. Section 4 is devoted to gravitational fields which can be all considered as different representations of the same nonlinear σ-model. We treat the case of stationary axisymmetric spacetimes, Einstein–Rosen gravitational waves, and Gowdy cosmological models of the type T^3 and $S^2 \times S^1$. In all these cases, we show that the physical fields of the $SL(2, R)/SO(2)$ nonlinear σ-model can be identified in different ways with the coefficients of the spacetime metric to generate different gravitational fields.

2. NONLINEAR σ-MODELS

A nonlinear σ-model is a field theory with the following properties (Balachandran *et al.*, 1990):

(i) The fields are subject to nonlinear constraints.

(ii) The Lagrangian L, or equivalently the Lagrangian density $\mathscr{L}$, and the constraints are invariant under the action of a global symmetry group G.

The description "nonlinear" refers to those models where the physical fields for all points x of the spacetime X take values in a manifold M which is not a vector space. In most of these models, the group G acts transitively on M. Therefore, if H is an invariant subgroup of G and also is the stability group of a point $p \in M$, then $M = G/H$.

Another basic part of a nonlinear σ-model is its Lagrangian density. In the standard approach, one demands that the fields of the Lagrangian density $\mathscr{L}$ take values in the Lie group G, i.e., $g(x) \in G$ (Balachandran *et al.*, 1990). Furthermore, $\mathscr{L}$ must be invariant under the gauge transformations where H is an invariant subgroup of G,

$$g(x) \to g(x)h(x), \qquad h(x) \in H \subset G \tag{1}$$

Therefore the gauge-invariant fields (or physical fields) take values in G/H.

Let G be a faithful representation of G and let $\{L_\rho\}$ be a basis for the Lie algebra $\mathscr{G}$ of G with the following property: $|Tr\{L_\rho L_\sigma\}| = \delta_{\rho\sigma}$, $\rho, \sigma = \{1, 2, \ldots, [G]\}$. For $\alpha \leq [H]$, the generators L_α are taken to span the Lie algebra $\mathscr{H}$ of H and we denote them by T_α. The remaining generators are called S_i with $[H] + 1 \leq i \leq [G]$. The commutation relations are

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta} T_\gamma, \qquad [T_\alpha, S_i] = i\overline{C}^j_{\alpha i} S_j, \qquad [S_i, S_j] = i\{D^\alpha_{ij} T_\alpha + \overline{D}^k_{ij} S_k\} \tag{2}$$

where $D^\alpha_{ij} = \overline{C}^j_{\alpha i}$ and the bar denotes complex conjugation.

Let w be the 1-form defined on G with components $w_\mu(g) = g^{-1}\partial_\mu g$ which under a gauge transformation of the form (1) transform as

$$h \in H, \qquad w_\mu(gh) = h^{-1}w_\mu(g)h + h^{-1}\partial_\mu h, \qquad w_\mu = A_\mu + B_\mu$$
$$A_\mu(g) = T_\alpha Tr\{T^\alpha w_\mu(g)\}, \qquad B_\mu(g) = S_i Tr\{S^i w_\mu(g)\}$$
$$A_\mu(gh) = h^{-1}\partial_\mu h + h^{-1}A_\mu(g)h, \qquad B_\mu(gh) = h^{-1}B_\mu(g)h$$

We have the structure of a principal fiber bundle with total space $E = G$, base space $B = G/H$, fiber $F \simeq H$, structure group H, and projector Π: $G \to G/H$, $g \mapsto [g]$. By construction, all fields lie in G and the gauge-invariant fields lie in the base space G/H. Due to this structure, we have then that the 1-form A with components A_μ transforms like a gauge potential (for the gauge group H) and therefore it acts as a connection 1-form. Thus from A we can construct the curvature 2-form F which under (1) transforms us $F' = h^{-1}Fh$.

Let $g^{\mu\nu}$ be the metric tensor of the spacetime X and g its determinant. The structure of the 1-form ω allows one to construct different quantities satisfying the above given requirements to be a Lagrangian density for a nonlinear σ-model,

$$\mathcal{L}_\sigma = \sqrt{-g}g^{\mu\nu}Tr\{B_\mu B_\nu\} \tag{3}$$
$$\mathcal{L}_\sigma = \sqrt{-g}g^{\mu\tau}g^{\nu\rho}Tr\{F_{\mu\nu}F_{\tau\rho}\}$$

Since the 1-form w is a section in the cotangent bundle T^*G, then $\mathcal{L}_\sigma$ is a function of one section in T^*G and is invariant under changes of sections in T^*G induced by the gauge group H. The changes induced in T^*G are the consequence of transformations $g \to gh$ on G, so $\mathcal{L}_\sigma$ is also invariant under changes of sections in G. Therefore we need to fix a section in G (fix a gauge) to find a Lagrangian density $\mathcal{L}_\sigma$. If Υ is a gauge-invariant field, then

$$\Upsilon = g(\sum_\alpha T_\alpha)g^{-1} \in \mathcal{G}, \qquad g(\sum_\alpha T_\alpha)g^{-1} = \sum_\rho v_\rho L_\rho \tag{4}$$

where the fields v_ρ are also physical fields. From (2) and (4), the following relationship must hold:

$$\sum_\alpha Tr\{T_\alpha T_\alpha\} = \sum_\rho v_\rho^2 Tr\{L_\rho L_\rho\} \tag{5}$$

It follows from (5) that the fields v_ρ are subject to nonlinear constraints and there are only $[G] - 1$ independent physical field. The manifold M is defined by the constraint (5). Due to the nonlinear character of this constraint, M is not a vector space.

3. $G = SL(2, R)$, $H = SO(2)$

The generators of $\mathcal{SL}(2, R)$ are given as

$$L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad L_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|Tr\{L_\rho L_\sigma\}| = \delta_{\rho\sigma} \qquad \text{with} \quad \rho, \sigma = \{1, 2, 3 = [SL(2, R)]\}$$

The generator of $\mathcal{SO}(2)$ is L_1. In this case, only two invariant gauge fields are independent, and from (5), we have

$$v_1^2 - (v_2^2 + v_3^2) = 1$$

As a consequence, it is natural to fix a gauge in G in terms of two independent fields $\eta = \eta(x^1, x^2)$ and $\xi = \xi(x^1, x^2)$ and fix the gauge as follows (Nicolai, 1991):

$$g(x^1, x^2) = \begin{pmatrix} e^{\eta} & \xi e^{-\eta} \\ 0 & e^{-\eta} \end{pmatrix} \in SL(2, R) \tag{6}$$

By construction, $g(x^1, x^2)$ is a section in $SL(2, R)$ through which we can specify the Lagrangian density $\mathcal{L}_\sigma$ for this σ-model. From (6), we get that the components of the 1-form w are given by

$$w_\mu(g) = g^{-1}\partial_\mu g = \begin{pmatrix} \eta_\mu & e^{-2\eta}\xi_\mu \\ 0 & -\eta_\mu \end{pmatrix}, \qquad \eta_\mu := \partial\eta/\partial x^\mu, \quad \xi_\mu := \partial\xi/\partial x^\mu \tag{7}$$

Since $B_\mu(g) = S_i\, Tr\{S^i w_\mu(g)\}$, we find that

$$B_\mu(g) = \begin{pmatrix} \eta_\mu & \frac{1}{2}e^{-2\eta}\xi_\mu \\ \frac{1}{2}e^{-2\eta}\xi_\mu & -\eta_\mu \end{pmatrix}, \qquad Tr\{B_\mu B_\nu\} = 2\eta_\mu\eta_\nu + \tfrac{1}{2}e^{-4\eta}\xi_\mu\xi_\nu$$

$$\mathcal{L}_\sigma = \sqrt{-g}\, g^{\mu\nu}\left(2\eta_\mu\eta_\nu + \tfrac{1}{2}e^{-4\eta}\xi_\mu\xi_\nu\right) \tag{8}$$

A Lagrangian density (8) is for the nonlinear σ-model with $G = SL(2, R)$ and $H = SO(2)$. The Euler–Lagrange field equations are

$$g^{\alpha\mu}\xi_{\alpha\mu} + \xi_\mu g^{\alpha\mu}_{,\alpha} - 4g^{\alpha\mu}\xi_\mu\eta_\alpha + \tfrac{1}{2}g^{\alpha\mu}\xi_\mu(ln|g|)_{,\alpha} = 0$$

$$g^{\alpha\mu}\eta_{\alpha\mu} + \eta_\mu g^{\alpha\mu}_{,\alpha} + \tfrac{1}{2}g^{\alpha\mu}\eta_\mu(ln|g|)_{,\alpha} + \tfrac{1}{2}e^{-4\eta}g^{\alpha\mu}\xi_\alpha\xi_\mu = 0$$

$$\eta_{\mu\nu} := \partial^2\eta/\partial x^\mu\partial x^\nu \qquad \text{and} \qquad \xi_{\mu\nu} := \partial^2\xi/\partial x^\mu\partial x^\nu$$

4. GRAVITATIONAL FIELDS

We will show examples of gravitational fields whose action can be expressed as the Lagrangian density for the $SL(2, R)/SO(2)$ nonlinear σ-model presented in the previous section.

4.1. The Stationary Axisymmetric Gravitational Field

The stationary axisymmetric gravitational field in Weyl canonical coordinates (Kramer *et al.*, 1980) is

$$ds^2 = e^{2\psi}(dt - \omega d\phi)^2 - e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \tag{9}$$

where ψ, ω, and γ are functions of ρ and z only. The calculation of the scalar curvature leads to the Einstein–Hilbert Lagrangian density

$$D := (\partial_\rho, \partial_z), \qquad \mathscr{L}_{ST} = 2D\rho D\gamma + \frac{e^{4\psi}}{2\rho}(D\omega)^2 - 2\rho(D\psi)^2 \tag{10}$$

where we have neglected the total divergence terms.

Since γ and ω are cyclic "coordinates" of the Lagrangian density (10), we can find the Routhian density for those coordinates

$$\mathscr{R}_{ST} := \frac{\partial \mathscr{L}_{ST}}{\partial D\gamma} D\gamma + \frac{\partial \mathscr{L}_{ST}}{\partial D\omega} D\omega - \mathscr{L}_{ST} = \tfrac{1}{2}\rho e^{-4\psi}\,\Pi_\omega^2 + 2\rho(D\psi)^2 \tag{11}$$

where $\Pi_\omega = \rho^{-1}e^{4\psi}D_\omega$ is the canonically conjugate "momentum" associated with the generalized "coordinate" ω. Since the momentum Π_ω satisfies the equation $D\Pi_\omega = 0$, one can introduce a function Ω of ρ and z such that $\tilde{D}\Omega := \Pi_\omega$, where $\tilde{D} := (-\partial_x, \partial_\rho)$. Then, the equation $D\Pi_\omega = 0$ becomes an identity once the integrability condition $D\tilde{D}\Omega = 0 \leftrightarrow \Omega_{\rho z} = \Omega_{z\rho}$ is satisfied. The final form of the Routhian density (11) from which the field equations can be derived is

$$\mathscr{R}_{ST} = 2\rho(D\psi)^2 + \frac{\rho}{2}e^{-4\psi}(D\Omega)^2 \tag{12}$$

Equation (12) is also the Lagrangian density for the nonlinear σ-model derived in the previous section. If we identify $\eta = \psi$ and $\xi = \Omega$ and introduce the components of the metric (9) into the Lagrangian density (8), we see that $\mathscr{R}_{ST} = \mathscr{L}_\sigma$. The field equations turn out to be

$$\rho(\psi_{\rho\rho} + \psi_{zz}) + \psi_\rho + \tfrac{1}{2}\rho e^{4\psi}(\Omega_\rho^2 + \Omega_z^2) = 0$$

$$\Omega_{\rho\rho} + \Omega_{zz} - 4(\psi_\rho\Omega_\rho + \psi_z\Omega_z) + \rho^{-1}\Omega_\rho = 0$$

From the definition Π_ω and Ω, it follows that $\Omega_\rho = \rho^{-1}e^{4\psi}\omega_z$ and $\Omega_z = -\rho^{-1}e^{4\psi}\omega_\rho$; therefore (4.1) is one the Einstein equations in Weyl canonical coordinates. From the identity (4.1) and from the integrability condition $\Omega_{\rho z} = \Omega_{z\rho}$, we get the second Einstein equation

$$\omega_{\rho\rho} + \omega_{zz} + 4(\psi_\rho\omega_\rho + \psi_z\omega_z) - \rho^{-1}\omega_\rho = 0$$

This ends the proof that stationary axisymmetric gravitational fields with the identification $\psi = \eta$ and $\Omega = \xi$ can be treated as a special case of the nonlinear σ-model with $G = SL(2, R)$ and $H = SO(2)$.

4.2. The Einstein–Rosen Field

The Einstein–Rosen line element is (Kramer *et al.*, 1980)

$$ds^2 = e^{2(\gamma-\psi)}\, dt^2 - e^{-2\psi}(e^{2\gamma}d\rho^2 + \rho^2 d\phi^2) - e^{2\psi}(dz + \omega d\phi)^2 \qquad (13)$$

where ψ, ω, and γ are functions of t and ρ. The reduced Einstein–Hilbert density and the Routhian density are given by

$$\mathcal{L}_{ER} = 2\rho(D\psi)^2 + \frac{1}{2\rho}\, e^{4\psi}(D\omega)^2 - 2D\rho D\gamma, \qquad D := (\partial_\rho, i\partial_t)$$

$$\mathcal{R}_{ER} = \frac{\partial\mathcal{L}_{ER}}{\partial D\gamma}\, D\gamma + \frac{\partial\mathcal{L}_{ER}}{\partial D\omega}\, D\omega - \mathcal{L}_{ER} = -2\rho(D\psi)^2 + \frac{\rho}{2}\, e^{-4\psi}\Pi_\omega^2$$

$$\Pi_\omega := \partial\mathcal{L}_{ER}/\partial D\omega = \rho^{-1}e^{4\psi}D\omega$$

The dynamical Einstein equations can be written as

$$D^2\psi + \rho^{-1}D\rho D\psi + \tfrac{1}{2}e^{-4\psi}\Pi_\omega^2 = 0, \qquad D\Pi_\omega = 0$$

We can rewrite the Routhian density by introducing a real function Ω such that $\Pi_\omega = i\tilde{D}\Omega$, where $\tilde{D} := (-i\partial_t, \partial_\rho)$,

$$\mathcal{R}_{ER} = -2\rho(D\psi)^2 - \tfrac{1}{2}\rho e^{-4\psi}(D\Omega)^2 = 2\rho(\psi_t{}^2 - \psi_\rho{}^2) + \tfrac{1}{2}\rho e^{-4\psi}(\Omega_t{}^2 - \Omega_\rho{}^2) \qquad (14)$$

and the field equations are given by

$$D^2\psi + \rho^{-1}D\rho D\psi - \tfrac{1}{2}e^{-4\psi}(\tilde{D}\Omega)^2 = 0$$

$$D\tilde{D}\Omega = 0 \qquad \text{or equivalently} \quad \Omega_{t\rho} = \Omega_{\rho t}$$

If the fields η and ξ depend on t and ρ only and preserve the gauge (6), then the Lagrangian density (8) for the metric (13) gives

$$\mathcal{L}_\sigma = 2\rho(\eta_t^2 - \eta_\rho^2) + \frac{\rho}{2}\, e^{-4\eta}(\xi_t^2 - \xi_\rho^2) \qquad (15)$$

The Routhian density (15) is (14) for the Einstein–Rosen line element under the identification $\eta = \psi$ and $\xi = \Omega$, where now $\Omega_t = \rho^{-1}e^{4\psi}\omega_\rho$ and $\Omega_\rho = \rho^{-1}e^{4\psi}\omega_t$.

4.3. The Unpolarized T^3 and $S^1 \times S^2$ Gowdy Cosmological Models

The unpolarized T^3 and $S^1 \times S^2$ cosmological model is

$$ds^2 = -e^{-(\lambda+3\tau)/2}\, d\tau^2 + e^{-(\lambda-\tau)/2}\, d\theta^2 + h_1(\tau)h_2(\theta)[e^P(d\sigma + Qd\delta)^2 + e^{-P}\, d\delta^2] \tag{16}$$

where λ, P, and Q depend on the coordinates τ and θ only, and $0 \leq \sigma, \delta, \theta \leq 2\pi$. For the T^3 case, $h_1(\tau) = e^{-\tau}$ and $h_2(\theta) = 1$, whereas for the $S^1 \times S^2$ case, $h_1(\tau) = \sin e^{-\tau}$ and $h_2(\theta) = \sin\theta$. In order to include the two cases in only one line element (16), we made the change $e^P \mapsto e^P \sin\theta$ in the metric given in Quevedo and Ryan, (2000a) for the $S^1 \times S^2$ case. The reduced Einstein–Hilbert Lagrangian density is

$$\begin{aligned}\mathcal{L}_{GOW} &= \tfrac{1}{2}e^{\tau}h_1h_2[(P_\tau^2 - e^{-2\tau}P_\theta^2) + e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)] \\ &\quad + \tfrac{1}{2}e^{\tau}(h_2h_{1\tau}\lambda_\tau - e^{-2\tau}h_1h_{2\theta}\lambda_\theta) + V(\tau,\theta) \qquad (17)\\ V(\tau,\theta) &= e^{-\tau}h_1h_2^{-1}h_{2\theta}^2 - e^{\tau}h_2h_{1\tau}(h_{1\tau}h_1^{-1} + 1)\end{aligned}$$

Since λ is a cyclic “coordinate” of the Lagrangian density (17) and the term $V(\tau, \theta)$ does not affect the field equations, we can carry out a Legendre transformation such that (17) becomes

$$\mathcal{L}_{GOW} = \tfrac{1}{2}e^{\tau}h_1h_2[(P_\tau^2 - e^{-2\tau}P_\theta^2) + e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)] \tag{18}$$

From (18), we obtain the field equations

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} + P_\tau(1 + h_1^{-1}h_{1\tau}) - e^{-2\tau}h_2^{-1}h_{2\theta}P_\theta - e^{-2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) = 0$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\theta\theta} + Q_\tau(1 + h_1^{-1}h_{1\tau}) - e^{-2\tau}h_2^{-1}h_{2\theta}Q_\theta + 2(P_\tau Q_\tau - e^{-2\tau}P_\theta Q_\theta) = 0$$

For the T^3 case ($h_1 = e^{-\tau}$, $h_2 = 1$), we require the gauge (6) and that the fields η and ξ depend on τ and θ only. Equation (3) yields a Lagrangian density

$$\mathcal{L}_\sigma = \tfrac{1}{2}[(4\eta_\tau^2 - 4e^{-2\tau}\eta_\theta^2) + e^{-4\eta}(\xi_\tau^2 - e^{-2\tau}\xi_\theta^2)] \tag{19}$$

which, after the identification $\eta = -P/2$ and $\xi = Q$, is identical to the Einstein–Hilbert Lagrangian (18) for the unpolarized T^3 Gowdy model.

If we take the metric (16) with $h_1(\tau) = \sin e^{-\tau}$ and $h_2(\theta) = \sin\theta$, impose that the fields η and ξ depend on τ and θ only, and preserve the same gauge (6), then $\mathcal{L}_\sigma$ takes the form

$$\mathcal{L}_\sigma = \tfrac{1}{2}e^{\tau}\sin e^{-\tau}\sin\theta[4\eta_\tau^2 - 4e^{-2\tau}\eta_\theta^2 + e^{-4\eta}(\xi_\tau^2 - e^{-2\tau}\xi_\theta^2)] \tag{20}$$

After the identification $\eta = -P/2$ and $\xi = Q$, the Lagrangian (20) coincides with the Einstein–Hilbert reduced Lagrangian (18) for the Gowdy cosmological model with topology $S^1 \times S^2$. The Lagrangian (20) allows us to derive an Ernst-like field equation for Gowdy cosmological models which can be used as starting point for applying solution-generating techniques. This is a novel result which will allow us to find exact solutions of this type and contradicts the general belief that exact solutions are difficult if not impossible to derive.

5. CONCLUSIONS

We have shown that the Einstein–Hilbert action for stationary axisymmetric space-times, Einstein–Rosen gravitational waves, and Gowdy cosmological models can be treated as the action for the $SL(2, R)/SO(2)$ σ-model. For each case, we have explicitly given the relationship between the physical fields of the nonlinear σ-model and the coefficients of the corresponding spacetime metric.

This result is the first step in a program which has as a goal to compare different quantization procedures for the same gravitational field. Einstein–Rosen waves have been investigated in the context of *nonperturbative* quantum gravity by using Ashtekar variables (Ashtekar *et al.*, 1996) with the result that the space of physical states can be contacted explicitly. We can also perform the *exact* quantization procedure for this spacetime by using the property of complete integrability for the corresponding $SL(2, R)/SO(2)$. The results would be similar to those obtained for stationary axisymmetric spacetimes. On the other hand, one could take the Lagrangian density for Einstein–Rosen waves and treat it as the starting point for applying the formalism of canonical quantization. Studies show that we obtain a field theory with constraints for which apparently the BRST quantization procedure would be useful. If succesful, the results of this quantization could be compared with the former procedures (exact and nonperturbative) in order to get some insight into the quantum structure of this system.

In the case of Gowdy cosmological models, the results of quantum cosmology, based on the "midisuperspace" quantization approach, can be applied to construct the space of physical quantum states (Quevedo and Ryan, 2000b). According to our results, the exact quantization procedure of the nonlinear $SL(2, R)/SO(2)$ σ-model can also be applied to this specific spacetime. The two approaches can then be investigated and compared.

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